

Stability of the splay state in networks of pulse-coupled neurons

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Abstract We analytically investigate the stability of *splay states* in networks of N pulse-coupled phase-like models of neurons. By developing a perturbative technique, we find that, in the limit of large N , the Floquet spectrum scales as $1/N^2$ for generic discontinuous velocity fields. Moreover, the stability of the so-called short-wavelength component is determined by the sign of the jump at the discontinuity. Altogether, the form of the spectrum depends on the pulse shape but is independent of the velocity field.

Keywords Pulse-coupled neural networks · Floquet spectra · Splay states

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1 Introduction

The first objective of a (neural) network theory is the identification of the asymptotic regimes. The last-decades activity have led to the discovery of fully- and partially-synchronized states, clusters and splay or asynchronous states in pulse-coupled networks [1,2,3,4]. It has also been made clear that ingredients such as disorder (diversity of the neurons and structure of the connections) are very important in determining the asymptotic behaviour, as well as the possible presence

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of delayed interactions and plasticity [5,6]. However, even if one restricts the analysis to identical, globally-coupled oscillators, there are very few theoretical results: they mostly concern fully synchronized regime or specific types of neurons (e.g. the leaky integrate-and-fire model) [4,7,8].

In this paper, we develop a perturbative analysis for the stability of *splay states* (also known as antiphase states [9], “ponies on a merry-go-round” [10], or rotating waves [11]) in ensembles of N identical fully pulse-coupled neurons. In a splay state, all the neurons follow the same periodic dynamics except for a time shift that is evenly distributed. Splay states have been identified in experimental measurements performed on electronic circuits [11] and on multimode lasers [12]. Theoretical studies have been devoted to splay states in fully coupled Ginzburg-Landau equations [13], Josephson arrays [14], laser models [15], traffic models [16], and pulse-coupled neuronal networks [2]. In the latter context, splay states have been mainly investigated in leaky-integrate-and-fire (LIF) neurons [2,3,17,18], but some studies have been also devoted to the θ -neurons [19] and to more realistic neuronal models [20]. Finally, splay states are important in that they provide the simplest instance of asynchronous behaviour and can be thereby used as a testing ground for the stability of a more general class of dynamical regimes.

Our model neurons are characterized by a membrane potential u that is continuously driven by the velocity field $F(u)$, from the resetting value $u = 0$ towards the threshold $u = 1$ (see the next section for a more precise definition). As threshold and resetting value can be identified with one another and thereby u interpreted as a phase, it will be customary to refer to the case $F(1) \neq F(0)$ as to that of a discontinuous velocity field. Additionally, we assume that the post-synaptic potential (PSP) has a stereotyped shape, the so-called α -pulse, that is characterized by an identical rise and decay time $1/\alpha$ [2]. As already discussed in [18], the Floquet spectrum is composed of two components: (i) long wavelengths (LWs), which can be studied in terms of a suitable functional equation for the probability distribution of the membrane potential u [2]; (ii) short-wavelengths (SWs), which typically correspond to marginally stable directions in the thermodynamic limit ($N \rightarrow \infty$). By developing an approach that is valid for arbitrary coupling strength and is perturbative in the inverse system-size $1/N$, we prove that the SW component of the Floquet spectrum scales as $1/N^2$ and is proportional to $F(1) - F(0)$, i.e. it is present only if the velocity field is discontinuous. We are also able to determine the spectral shape and find it to be *universal*, i.e. independent of the details of the velocity field.

More precisely, we first build the corresponding event-driven map, by expanding it in powers of $1/N$ (a posteriori, we have verified that it is necessary to reach the fourth order). Afterwards, the expression of the splay state is determined: this task corresponds to finding a fixed point of the event-driven map in a suitably moving reference frame - analogously to what previously done in specific contexts [18,21,22]. In practice this task is carried out by first taking the continuum limit for the various orders and obtaining suitable differential equations, whose solution allows proving that all finite-size corrections for both the period T and the membrane potential vanish up to the third order. Next, the stability analysis is carried out to determine the leading term to the Floquet spectrum. This task involves the introduction of a suitable Ansatz to decompose each eigenvector into the linear superposition of a slow and a rapidly oscillating component. The following contin-

uum limit shows that the two components satisfy an ordinary and a differential equation, respectively.

Altogether, the proof of our main result requires determining all terms up to the third order in the $1/N$ expansion of the splay state solution, while some third order terms are not necessary for the tangent space analysis. Going beyond discontinuous fields would require extending our analysis to account for higher order terms and this might not even be sufficient to characterize analytic velocity fields. In fact, previous numerical simulations [21] suggest that the Floquet exponents scale with higher powers of $1/N$ that depend on which derivatives of $F(u)$ are eventually discontinuous. Moreover, it is worth recalling that in the case of a strictly sinusoidal field, a theorem proved by Watanabe and Strogatz [23] implies that $N - 3$ Floquet exponents ($N - 2$ for a splay solution) vanish exactly for any value of N .

In the small coupling limit, one can combine our results with those of Abbott and van Vreeswijk [2] (that are valid only in that regime) for the LW spectral component and conclude that the splay state is stable whenever $F(0) > F(1)$ and the pulses are sufficiently broad, for excitatory coupling, while it is always unstable for inhibitory coupling and any finite pulse-width. This scenario is partially reminiscent of the stability of synchronous and clustered regimes that is determined by the sign of the first derivative dF/du of the velocity-field averaged on the interval $[0, 1]$ (this latter problem has been investigated in excitatory pulse-coupled integrate-and-fire oscillators subject to δ -pulses [1, 24]).

Section II is devoted to the introduction of the model and to a brief presentation of the main results, including an expression for the leading correction to the period for the LIF model, to provide evidence that they are typically of 4th order. A general perturbative expression for the map is derived in Sec. III, while Sec. IV is devoted to deriving the splay-state solution up to the third order in $1/N$. The main result of the paper is discussed in Sect. V, where the Floquet spectra are finally obtained. Sect. VI contains some general remarks and a discussion of the open problems. The technical details of some lengthy calculations have been confined in the appendices: Appendix A is devoted to the derivation of the splay state solution; Appendix B contains the derivation of the leading term (of order four) of the period T for the LIF model; Appendix C is concerned with the linear stability analysis.

2 Model and main results

We consider a network of N identical neurons (rotators) coupled via a mean-field term. The dynamics of the i -th neuron writes as

$$\dot{u}_i(t) = F(u_i) + gE(t) \equiv \mathcal{F}_i(t) \quad i = 1, \dots, N, \quad (1)$$

where $u_i(t)$ represents the membrane potential, $E(t)$ is the forcing field, and g is the coupling constant. When the membrane potential reaches the threshold value $u_i(t) = 1$, a spike is sent to all neurons (see below for the relationship between the single spikes and the global forcing field $E(t)$) and it is reset to $u_i(t) = 0$. The resetting procedure is an approximate way to describe the discharge mechanism operating in real neurons. The function F represents a velocity field for the isolated neuron and it is assumed to be everywhere positive (thus ensuring that the neurons repetitively fire, since they are supra-threshold), while \mathcal{F}_i is the velocity field seen

by the neuron i in the presence of a coupling with other neurons. While we consider both excitatory ($g > 0$) and inhibitory networks ($g < 0$), it is easy to show that \mathcal{F} remains always positive to ensure the existence of splay states. For the simple choice

$$F(u) = a - u \quad , \quad (2)$$

the model reduces to the well known case of LIF neurons.

The field E is the linear superposition of the pulses emitted in the past when the membrane potential of each single neuron has reached the threshold value. By following Ref. [2], we assume that the shape of a pulse emitted at time $t = 0$ is given by $E_s(t) = \frac{\alpha^2 t}{N} e^{-\alpha t}$, where $1/\alpha$ is the pulse-width. This is equivalent to saying that the total field evolves according to the equation

$$\ddot{E}(t) + 2\alpha\dot{E}(t) + \alpha^2 E(t) = \frac{\alpha^2}{N} \sum_{n|t_n < t} \delta(t - t_n) . \quad (3)$$

where the sum in the r.h.s. represents the source term due to the spikes emitted at times $t_n < t$.

It is convenient to transform the continuous-time model into a discrete-time mapping. We do so by integrating the equations of motion from time t_n to time t_{n+1} (where t_n is the time immediately after the n -th pulse has been emitted). The resulting map for the field variables reads,

$$\begin{aligned} E_{n+1} &= [E_n + \tau_n P_n] e^{-\alpha \tau_n} , \\ P_{n+1} &= P_n e^{-\alpha \tau_n} + \frac{\alpha^2}{N} , \end{aligned} \quad (4)$$

where $\tau_n = t_{n+1} - t_n$ is the interspike time interval and, for the sake of simplicity, we have introduced the new variable $P := \alpha E + \dot{E}$.

In this paper we focus on a specific solution of the networks dynamics, namely on *splay states*, which are asynchronous states, where all neurons fire periodically with period T and two successive spike emissions occur at regular intervals $\tau_n \equiv T/N$. The first result of this paper is that under the assumption that the velocity field $F(u)$ is differentiable at least four times, the dependence of the period T onto the size N is of order $o(1/N^3)$. In the specific case of LIF neurons, we show in Appendix B that the leading correction δT to the infinite size result is indeed of order $O(1/N^4)$ and, more precisely,

$$\delta T = \frac{K(\alpha) - 6}{720} \frac{[a(1 - e^{-T}) - 1]}{ge^{-T} + a(T + 1 - e^{-T}) - 1} \frac{T^5}{N^4} , \quad (5)$$

where $K(\alpha)$ encodes the information on the pulse dynamics (see Eq. (66)). We did not dare to estimate the quartic contribution for generic velocity fields, not only because the algebra would be utterly complicated, but also since our main motivation is to determine the leading contributions in the stability analysis, and it turns out that it is sufficient to determine the splay state up to the third order.

The study of the stability requires determining the Floquet spectrum, i.e. the complex eigenvalues of a given periodic orbit of period T . With reference to a system of size N , the Floquet multipliers can be written as

$$\mu_k = e^{i\phi_k} e^{(\lambda_k + i\omega_k)\tau_n} , \quad k = 1, \dots, 3N , \quad (6)$$

where ϕ_k represents the 0th order phase (that is responsible for the high frequency oscillations of the corresponding eigenvector - see Sec. V), while λ_k and ω_k are the real and imaginary parts of the Floquet exponent, respectively. In the following we prove that the leading term of the SW component (i.e. for ϕ_k away from zero), is

$$\lambda_k = \frac{g\alpha^2}{12} \frac{F(1) - F(0)}{\mathcal{F}(1)\mathcal{F}(0)} \left(\frac{6}{1 - \cos \phi_k} - 1 \right) \frac{1}{N^2} . \quad (7)$$

For discontinuous velocity fields, the real parts of the spectrum scale as $1/N^2$, while the imaginary parts are of even higher order.

For continuous fields, it has been numerically observed that the scaling of the spectrum is at least $O(1/N^4)$ [21]. In other words the shape of the spectrum is universal, apart from a multiplicative factor that vanishes if and only if $F(1) = F(0)$, i.e. for true phase rotators where $u = 0$ coincides with $u = 1$. The stability of the splay state can be inferred by the sign of $F(1) - F(0)$: in the case of excitatory (resp. inhibitory) coupling, the state is stable whenever $F(0) > F(1)$ (resp. $F(0) < F(1)$). In the limit $\phi_k \rightarrow 0$ the expression reported in parenthesis in Eq. (7) diverges, indicating that the perturbative analysis breaks down. This limit corresponds to the LW component, where our approach can be complemented by that of Abbott and van Vreeswijk [2], which reveals that the corresponding Floquet exponents do not depend on the system size. For sufficiently small couplings ($|g| \ll 1$), they also found a condition similar to the one reported above, namely that, irrespectively of the sign of the coupling, the splay state is stable whenever $F(0) > F(1)$ for sufficiently broad pulses. In fact, above a critical α -value (i.e. below a given pulsewidth), the splay state loses stability due to a supercritical Hopf bifurcation, which leads to the emergence of a more complex collective regime, termed *partial synchronization* [3, 25]. By combining the conditions for the SW and the LW spectrum, one can predict the overall stability of the splay state. In particular, the state is stable for excitatory coupling if $F(0) > F(1)$ (and α sufficiently small), while it is always unstable for finite networks, for inhibitory coupling, since the SW and LW stability conditions are opposite to one another. This last result is consistent with the findings reported by van Vreeswijk for inhibitory coupling and finite pulse width [3].

3 Event driven map

By following Ref. [26, 21], it is convenient to pass from a continuous to a discrete time evolution rule, by introducing the event-driven map which connects the network configuration at subsequent spike emissions occurring at time t_n and t_{n+1} . The membrane-potential value $u_i(t_{n+1}^-)$ just before the emission of the $(n+1)$ -th spike can be obtained by formally integrating Eq. (1),

$$u_i(t_{n+1}^-) - u_{n,i}(t_n) = \int_{t_n}^{t_{n+1}^-} dt F(u_i(t)) + g \int_{t_n}^{t_{n+1}^-} dt [E_n + P_n(t - t_n)] e^{-\alpha t} \equiv \mathcal{A}_1 + \mathcal{A}_2, \quad (8)$$

where the minus superscript means that the map construction has not yet been completed. This task is accomplished by ordering the membrane potentials from the largest ($j = 1$) to the smallest value ($j = N$) value and by passing to a

comoving frame that advances with the firing neuron, i.e. by shifting the neuron index by one unit,

$$u_{n+1,j-1} = u_j(t_{n+1}^-), \quad (9)$$

where the first subscript indicates that the variable is determined at time t_{n+1} . This change of reference frame allows treating the splay state as a fixed point of the event driven map.

The first integral appearing on the rhs of Eq. (8) is now solved perturbatively by introducing a polynomial expansion of $u_i(t)$ around $t = t_n$, which, up to third order, reads as

$$u_j(t) = u_{n,j} + \dot{u}_{n,j}\delta t + \frac{1}{2}\ddot{u}_{n,j}\delta t^2 + \frac{1}{6}\dddot{u}_{n,j}\delta t^3 + O(\delta t^4), \quad (10)$$

where $\delta t = t - t_n$. Explicit expressions for the time derivatives of u_j can be obtained from Eq. (1) and its time derivatives,

$$\begin{aligned} \ddot{u}_{n,j} &= F'(u_{n,j})\dot{u}_{n,j} + g\dot{E}_n, \\ \dddot{u}_{n,j} &= F''(u_{n,j})\dot{u}_{n,j}^2 + F'(u_{n,j})\ddot{u}_{n,j} + g\ddot{E}_n, \end{aligned}$$

where one can further eliminate \ddot{E}_n with the help of Eq. (3).

By inserting the expansion (10) into the expression of \mathcal{A}_1 , expanding the function $F(u)$, and performing the trivial integrations, one obtains

$$\begin{aligned} \mathcal{A}_1 &= F_{n,j}\tau_n + F'_{n,j}\mathcal{F}_{n,j}\frac{\tau_n^2}{2} + \left\{ \left[F''_{n,j}\mathcal{F}_{n,j} + F_{n,j}^{\prime 2} \right] \mathcal{F}_{n,j} + g\dot{E}_n F'_{n,j} \right\} \frac{\tau_n^3}{6} + \left\{ F_{n,j}^{\prime \prime \prime} \mathcal{F}_{n,j}^3 \right. \\ &\quad \left. + 4F'_{n,j}F''_{n,j}\mathcal{F}_{n,j}^2 + F_{n,j}^{\prime 3}\mathcal{F}_{n,j} + g \left[\left(3F''_{n,j}\mathcal{F}_{n,j} + F_{n,j}^{\prime 2} - \alpha F'_{n,j} \right) \dot{E}_n - \alpha F'_{n,j}P_n \right] \right\} \frac{\tau_n^4}{24} + O(\tau_n^5), \end{aligned} \quad (11)$$

where $\tau_n = t_{n+1} - t_n$ and we have introduced the short-hand notation $F_{n,j}$ for $F(u_{n,j})$ (and analogously for \mathcal{F}).

The explicit expression of \mathcal{A}_2 reads

$$\begin{aligned} \mathcal{A}_2 &= \frac{g}{\alpha}E_n(1 - e^{-\alpha\tau_n}) - \frac{g}{\alpha}P_n\tau_n e^{-\alpha\tau_n} + \frac{g}{\alpha^2}P_n(1 - e^{-\alpha\tau_n}) \\ &= gE_n\tau_n + g\dot{E}_n\frac{\tau_n^2}{2} - g\alpha\left(\dot{E}_n + P_n\right)\frac{\tau_n^3}{6} + g\alpha^2\left(\dot{E}_n + 2P_n\right)\frac{\tau_n^4}{24} + O(\tau_n^5). \end{aligned} \quad (12)$$

Now, by assembling Eqs. (8,9,11,12), we obtain the final expression for the evolution rule of the membrane potential,

$$\begin{aligned} u_{n+1,j-1} &= u_{n,j} + \mathcal{F}_{n,j}\tau_n + \left[g\dot{E}_n + F'_{n,j}\mathcal{F}_{n,j} \right] \frac{\tau_n^2}{2} + \left\{ F'_{n,j} \left[F'_{n,j}\mathcal{F}_{n,j} + g\dot{E}_n \right] \right. \\ &\quad \left. + F_{n,j}^{\prime \prime} \mathcal{F}_{n,j}^2 - g\alpha \left[P_n + \dot{E}_n \right] \right\} \frac{\tau_n^3}{6} + \left\{ -g\alpha \left(\dot{E}_n + P_n \right) F'_{n,j} + 4F'_{n,j}F''_{n,j}\mathcal{F}_{n,j}^2 \right. \\ &\quad \left. + F_{n,j}^{\prime 3}\mathcal{F}_{n,j} + g \left[\left(3F''_{n,j}\mathcal{F}_{n,j} + F_{n,j}^{\prime 2} \right) \dot{E}_n + \alpha^2 \left(\dot{E}_n + 2P_n \right) \right] + F_{n,j}^{\prime \prime \prime} \mathcal{F}_{n,j}^3 \right\} \frac{\tau_n^4}{24} + O(\tau_n^5). \end{aligned} \quad (13)$$

Eqs. (4) and (13) define the map we are going to investigate in the following sections. The time needed to reach the threshold τ_n can be determined implicitly from Eq. (13) by setting $j = 1$, since by definition of the model $u_{n,0} \equiv 1$.

4 Splay state solution

The splay state is a fixed point of the previous mapping corresponding to a constant interspike interval $\tau = T/N$. Since the fixed point solutions do not depend on the index n they are denoted as,

$$E_n \equiv \tilde{E} , \quad P_n \equiv \tilde{P} , \quad u_{n,j} \equiv \tilde{u}_j . \quad (14)$$

In order to study the dependence of the splay state on the system size N , it is necessary and sufficient to formally expand the expression of the membrane potentials as follows

$$\tilde{u}_j = \sum_{h=0,4} \frac{\tilde{u}_j^{(h)}}{N^h} + O\left(\frac{1}{N^5}\right) \quad (15)$$

and, analogously, for the period T ,

$$T = \sum_{h=0,4} \frac{T^{(h)}}{N^h} + O\left(\frac{1}{N^5}\right) . \quad (16)$$

This expansion can be performed by exploiting the explicit dependence of E and P on N , as detailed in Eqs. (56,55) in Appendix A.

Finally, by substituting the expressions (16,55,56,57) in Eq. (13) one obtains the evolution equations for the membrane potentials

$$\sum_{h=0,4} \frac{\tilde{u}_{j-1}^{(h)} - \tilde{u}_j^{(h)}}{N^h} = \sum_{h=1,4} \frac{Q^{(h)}}{N^h} + O\left(\frac{1}{N^5}\right) , \quad (17)$$

where the Q variables are defined in Appendix A.

In the large N limit, one can introduce the continuous spatial coordinate $x = j/N$. In practice, this is tantamount to write,

$$U^{(h)}(x = j/N) = \tilde{u}_j^{(h)} , \quad h = 0, \dots, 4 . \quad (18)$$

It is important to stress that the event-driven neuronal evolution in the comoving frame implies that $U(0) = 1$, i.e. the first neuron will fire at the next step, and $U(1) = 0$, i.e. the membrane potential of the last neuron has been just reset to zero. This implies that $U^{(0)}(0) = 1$ and $U^{(0)}(1) = 0$, while $U^{(h)}(0) = U^{(h)}(1) = 0$ for any $h > 0$.

Furthermore, by expanding $U^{(h)}(x)$ around $x = j/N$, one obtains

$$\tilde{u}_{j-1}^{(h)} = U^{(h)}(x - 1/N) = U^{(h)}(x) + \sum_{m=1,4} \frac{1}{m!} \left(\frac{-1}{N}\right)^m \frac{d^m}{dx^m} U^{(h)}(x) + O\left(\frac{1}{N^5}\right) , \quad (19)$$

By inserting this expansion into Eq. (17), we obtain an equation that can be effectively split into terms of different order that will be analysed separately. Notice that by retaining terms of order h , it is possible to determine the original variables at order $h - 1$.

4.1 Zeroth order approximation

By assembling the first order terms, we obtain the evolution equation for the zeroth order membrane potential, namely

$$\frac{dU^{(0)}}{dx} = -g - T^{(0)} F(U^{(0)}) \quad . \quad (20)$$

This equation is equal to the evolution equation of the membrane potential for a constant field E , with x playing the role of a (inverse) time. Please notice that, up to first order, $\tilde{E} = 1/T^{(0)}$ (see Eq. (56)). An implicit and formal solution of Eq. (20) is,

$$1 - x = \int_0^{U^{(0)}} \frac{dv}{g + T^{(0)} F(v)} \quad , \quad (21)$$

where we have imposed the condition $U^{(0)}(1) = 0$. However, there is a second condition to impose, namely $U^{(0)}(0) = 1$. This second condition transforms itself in the equation defining the interspike time interval $T^{(0)}$, when $N \rightarrow \infty$ (i.e. in the thermodynamic limit)

$$1 = \int_0^1 \frac{dU^{(0)}}{g + T^{(0)} F(U^{(0)})} \quad . \quad (22)$$

This result is, so far, quite standard and could have been easily obtained by just assuming a constant field E in equation (1). If we introduce the formal relation $F'[U^{(0)}(x)] = \frac{dF(U^{(0)})}{dU^{(0)}}$ in Eq. (20) we obtain

$$\frac{dF(U^{(0)})}{g + T^{(0)} F(U^{(0)})} = -F'[U^{(0)}(x)] dx \quad , \quad (23)$$

which can be easily integrated

$$\int_{F(U^{(0)}(0))}^{F(U^{(0)}(1))} \frac{dF(U^{(0)})}{g + T^{(0)} F(U^{(0)})} = - \int_0^1 F'[U^{(0)}(x)] dx \quad , \quad (24)$$

giving the following relation (already derived in [25], by following a different approach)

$$\frac{e^{-T^{(0)} H(0)}}{\mathcal{F}(U(0))} = \frac{e^{-T^{(0)} H(1)}}{\mathcal{F}(U(1))} \quad , \quad (25)$$

where, for later convenience, we have introduced

$$H(x) = \int_0^x F'[U^{(0)}(y)] dy \quad , \quad (26)$$

and where, for the sake of simplicity, the prime denotes derivative with respect to the variable $U^{(0)}$ and the dependence of F and F' on $U^{(0)}$ has been dropped.

4.2 First order approximation

By collecting the terms of order $1/N^2$, one obtains

$$\frac{dU^{(1)}}{dx} = -T^{(0)}F'U^{(1)} + \frac{1}{2}\frac{d^2U^{(0)}}{dx^2} - FT^{(1)} - \frac{1}{2}(T^{(0)})^2F'F - \frac{g}{2}T^{(0)}F' \quad . \quad (27)$$

An explicit expression for the second derivative of $U^{(0)}(x)$ appearing in Eq. (27) can be computed by deriving Eq. (20) with respect to x . This allows rewriting Eq. (27) in a simplified form, namely

$$\frac{dU^{(1)}(x)}{dx} = -U^{(1)}T^{(0)}F' - T^{(1)}F. \quad (28)$$

By imposing $U^{(1)}(1) = 0$, one obtains the general solution of Eq. (28),

$$U^{(1)}(x) = \int_x^1 du \, T^{(1)}F[U^{(0)}(u)] \exp \left[T^{(0)}(H(x) - H(u)) \right] \quad , \quad (29)$$

where $H(x)$ is defined by Eq. (26). The further condition to be satisfied, $U^{(1)}(0) = 0$, implies $T^{(1)} = 0$ and thereby we have $U^{(1)}(x) \equiv 0$, i.e. first-order corrections vanish both for the period and the membrane potential.

4.3 Second order approximation

The second order corrections can be estimated by assembling terms of order $1/N^3$ and by imposing the previously determined conditions $T^{(1)} = 0$ and $U^{(1)}(x) = 0$,

$$\begin{aligned} \frac{dU^{(2)}}{dx} = & -T^{(0)}F'U^{(2)} - \frac{1}{6}\frac{d^3U^{(0)}}{dx^3} - FT^{(2)} - \frac{g^2}{6}T^{(0)}F'' \\ & - \frac{g}{6}(T^{(0)})^2[2FF'' + F'^2] - \frac{(T^{(0)})^3}{6}[F''F^2 + F'^2F] \quad . \end{aligned}$$

Once evaluated $d^3U^{(0)}/dx^3$ from Eq. (20), the above ODE reduces to

$$\frac{dU^{(2)}}{dx} = -U^{(2)}T^{(0)}F' - T^{(2)}F \quad ; \quad (30)$$

which has the same structure as Eq. (28). Since one has also to impose the same boundary conditions as for the first order, namely $U^{(2)}(0) = U^{(2)}(1) = 0$, we can conclude that $T^{(2)} = 0$ and, consequently, $U^{(2)}(x) \equiv 0$. Therefore, second order corrections are absent too.

4.4 Third order approximation

By assembling terms of order $1/N^4$, once imposed that first and second order corrections vanish, one obtains

$$\begin{aligned} \frac{dU^{(3)}}{dx} = & -T^{(0)}F'U^{(3)} + \frac{1}{24}\frac{d^4U^{(0)}}{dx^4} - FT^{(3)} - \frac{g^3}{24}T^{(0)}F''' - \frac{g^2}{6}(T^{(0)})^2F'F'' \\ & - \frac{g^2}{8}(T^{(0)})^2FF''' - \frac{g}{24}(T^{(0)})^3[F'^3 + 8FF'F'' + 3gF^2F'''] \\ & - \frac{(T^{(0)})^4}{24}FF'[F'^2 + 4FF'' + F^3F'''] \quad . \end{aligned} \quad (31)$$

By replacing $d^4U^{(0)}/dx^4$ with its expression derived from Eq. (20), equation (31) takes the same form as in the two previous examined cases, namely

$$\frac{dU^{(3)}}{dx} = -U^{(3)}T^{(0)}F' - T^{(3)}F. \quad (32)$$

Therefore, we can safely conclude that third order terms vanish too.

The LIF model can be solved exactly for any value of N , starting from the asymptotic value ($N \rightarrow \infty$). As shown in Appendix B, it turns out that the leading corrections are of fourth order for both the period T and the membrane potential.

5 Linear stability analysis

The fixed-point analysis has revealed that the finite-size corrections to the stationary solutions are of order $o(1/N^3)$. Since such deviations do not affect the leading terms of the linear stability analysis (as it can be verified *a posteriori*) they will be simply neglected. Therefore, for the sake of simplicity, from now on, $T^{(0)}$ and $\tilde{u}_j^{(0)}$ will be simply referred to as T and \tilde{u}_j .

The evolution rule in tangent space is obtained by differentiating Eq. (13) and Eq. (4) around the fixed point solution. The explicit expression of the corresponding event-driven map is reported in Appendix C. It consists of evolution equations for δP_n and δE_n (Eqs. (77) and (78)), and for $\delta u_{n,j}$ (Eq. (79)). Finally, $\delta \tau_n$ is determined from Eq. (80).

As usual, the eigenvalue problem can be solved by introducing the Ansatz,

$$\delta u_{n,j} = \mu_k^n \delta u_j \quad \delta P_n = \mu_k^n \delta P \quad \delta E_n = \mu_k^n \delta E \quad \delta \tau_n = \mu_k^n \delta \tau, \quad (33)$$

where μ_k labels the eigenvalues, which must also be expanded as,

$$\mu_k = e^{i\phi_k} e^{(\lambda_k + i\omega_k)T/N} = e^{i\phi_k} \left(1 + \sum_{h=1,3} \frac{\Gamma^{(h)}}{N^h} + O\left(\frac{1}{N^4}\right) \right). \quad (34)$$

where $\Gamma^{(h)}$ is, in principle, a complex number and, for the sake of simplicity, we have dropped its dependence on k . Finally, as already shown, at zeroth order, the eigenvalues correspond to a pure rotation (specified by ϕ_k) with no expansion or contraction, i.e. $\Gamma^{(0)} = 0$.

By inserting the above Ansätze in the map expression (77,78,79,80), one obtains, after eliminating δP , δE and $\delta \tau$, a closed equation for δu_j ,

$$\begin{aligned} e^{i\phi_k} \left(1 + \frac{\Gamma^{(1)}}{N} + \frac{\Gamma^{(2)}}{N^2} + \frac{\Gamma^{(3)}}{N^3} \right) \delta u_{j-1} = & \left\{ 1 + \overline{F}'_j \frac{T}{N} + \left[\overline{F}''_j \overline{\mathcal{F}}_j + \overline{F}''^2_j \right] \frac{T^2}{2N^2} \right. \\ & + \left[\overline{F}'''_j \overline{\mathcal{F}}_j^2 + 4\overline{F}'_j \overline{F}''_j \overline{\mathcal{F}}_j + \overline{F}'''_j \right] \frac{T^3}{6N^3} \left. \right\} \delta u_j - \left\{ \overline{\mathcal{F}}_j + \overline{F}'_j \overline{\mathcal{F}}_j \frac{T}{N} + \left[\frac{\overline{F}''_j \overline{\mathcal{F}}_j^2}{2} + \frac{\overline{F}''^2_j}{2} \overline{\mathcal{F}}_j \right. \right. \\ & + \frac{g}{T} \alpha^2 \frac{e^{2i\phi_k} + 10e^{i\phi_k} + 1}{12(e^{i\phi_k} - 1)^2} \left(\frac{\overline{\mathcal{F}}_j}{\overline{\mathcal{F}}_1} - 1 \right) \left. \right] \frac{T^2}{N^2} + \left[\frac{\overline{F}'''_j}{6} \overline{\mathcal{F}}_j + \frac{2}{3} \overline{F}'_j \overline{F}''_j \overline{\mathcal{F}}_j^2 + \frac{\overline{F}'''_j}{6} \overline{\mathcal{F}}_j^3 \right. \\ & + \frac{g\alpha^2}{3T^2} \Gamma^{(1)} \frac{2e^{2i\phi_k} - 3e^{i\phi_k}}{(e^{i\phi_k} - 1)^3} \left(\frac{\overline{\mathcal{F}}_j}{\overline{\mathcal{F}}_1} - 1 \right) + \frac{g}{T} \alpha^2 \frac{e^{2i\phi_k} + 10e^{i\phi_k} + 1}{12(e^{i\phi_k} - 1)^2} \frac{\overline{\mathcal{F}}_j}{\overline{\mathcal{F}}_1} (\overline{F}'_j - \overline{F}'_1) \\ & \left. \left. + \frac{g\alpha^2}{T} \frac{5e^{i\phi} + 1}{12(e^{i\phi} - 1)^2} \left(\overline{F}'_1 \frac{\overline{\mathcal{F}}_j}{\overline{\mathcal{F}}_1} - \overline{F}'_j \right) + \frac{g\alpha^3}{T} \frac{e^{i\phi}(e^{i\phi} + 1)}{(e^{i\phi} - 1)^3} \left(1 - \frac{\overline{\mathcal{F}}_j}{\overline{\mathcal{F}}_1} \right) \right] \frac{T^3}{N^3} \right\} \frac{\delta u_1}{\overline{\mathcal{F}}_1} \quad , \end{aligned} \quad (35)$$

that is the object of our investigation. The overline means that the function is evaluated in $\tilde{u}_j^{(0)}$, corresponding to the infinite N limit.

5.1 Continuum limit

Similarly to the splay-state estimation, it is convenient to take the continuum limit. However, at variance with the previous case, now one should take in to account also the presence of fast scales associated to the “spatial” dependence of ϕ_k .

Therefore, the correct Ansatz is slightly more complicated and we have to separate slowly and rapidly oscillating terms,

$$\delta u_j = \pi_j + \vartheta_j e^{i\phi_k j}, \quad (36)$$

where the complex exponential term accounts for the fast oscillations of the eigenvectors, while,

$$\pi_j = \sum_{h=0,3} \frac{\pi_j^{(h)}}{N^h} + O\left(\frac{1}{N^4}\right) \quad , \quad \vartheta_j = \sum_{h=0,3} \frac{\vartheta_j^{(h)}}{N^h} + O\left(\frac{1}{N^4}\right) \quad , \quad (37)$$

are slowly varying variables.

Now, we can finally introduce the continuous variable $x = j/N$, as previously done in real space (see Eq. (18)),

$$\Pi_j^{(h)}(x = \frac{j}{N}) = \pi_j^{(h)} \quad , \quad \Theta_j^{(h)}(x = \frac{j}{N}) = \vartheta_j^{(h)} \quad , \quad (38)$$

where $h = 0, \dots, 3$. This allows expanding $\delta u_{j-1} = \pi_{j-1} + \vartheta_{j-1} e^{i\phi_k(j-1)}$ around $x = j/N$, similarly to what done in Eq. (19). At variance with the computation of the fixed point, now there are also terms like $U(1/N)$ and $\delta U(1/N)$, whose computation requires a similar expansion but around $x = 0$. By incorporating all the expansion terms within Eq. (35), we have finally an equation, where terms of different orders are naturally separated from one another. The calculations are summarized in Appendix C and the final equation is (85). By separately treating

the different orders, we obtain differential and ordinary equations for the Θ and Π variables. It turns out that it is necessary to consider in parallel different orders in the fast and slow terms to obtain Θ and Π to the same order. As a consequence, we will see that it is sufficient to expand $\delta U(1/N)$ up to order $O(1/N^3)$.

5.2 Zeroth order approximation

By assembling terms of order $\mathcal{O}(1/N)$ in Eq. (85), multiplied by the fast oscillating factor $e^{i\phi_k J}$, we obtain a first-order linear differential equation for $\Theta^{(0)}$, namely

$$\frac{d\Theta^{(0)}}{dx} = -\Theta^{(0)}(TF'(U(x)) - \Gamma^{(1)}), \quad (39)$$

where $\Gamma^{(1)}$ is the first order correction to the Floquet exponent which should be determined. It is important to remind that the prime denotes derivative with respect to the variable $U^{(0)}$, which has been simply redifined U , as previously mentioned. The solution is

$$\Theta^{(0)}(x) = K^{(0)} \exp \left[\Gamma^{(1)}x - TH(x) \right], \quad (40)$$

where we made use of the definition (26) and $K^{(0)}$ is a suitable integration constant.

By assembling now the slow terms of zeroth order and reminding the definition of $\mathcal{F}(U(x))$, we find the following algebraic equation

$$\Pi^{(0)}(x)(e^{i\phi} - 1) = - \left[e^{i\phi} \Theta^{(0)}(0) + \Pi^{(0)}(0) \right] \frac{\mathcal{F}(U(x))}{\mathcal{F}(U(0))} \quad . \quad (41)$$

With the help of Eq. (40), we obtain

$$\begin{aligned} \Pi^{(0)}(0) &= -\Theta^{(0)}(0) = -K^{(0)}e^{-TH(0)} \quad , \\ \Pi^{(0)}(x) &= -K^{(0)}e^{-TH(0)} \frac{\mathcal{F}(U(x))}{\mathcal{F}(U(0))} \quad . \end{aligned}$$

We can now impose the boundary condition $\delta U^{(0)}(x=1) = \Theta^{(0)}(1) + \Pi^{(0)}(1) = 0$. This implies that

$$\frac{e^{-TH(1)+\Gamma^{(1)}}}{\mathcal{F}(U(1))} = \frac{e^{-TH(0)}}{\mathcal{F}(U(0))} \quad . \quad (42)$$

By now exploiting Eq. (25), we find that $\Gamma^{(1)} = 0$, i.e. the Floquet exponent (both its real and its imaginary part) is equal to zero at first order in $1/N$. Furthermore, Eq. (40) becomes

$$\Theta^{(0)}(x) = K^{(0)}e^{-TH(x)}, \quad (43)$$

i.e. the eigenvectors are independent of the phase ϕ_k and are thereby equal to one another. In other words we are confirmed that the degeneracy has not been removed.

5.3 First order approximation

By assembling the fast terms of order $1/N^2$ and by setting $\Gamma^{(1)} = 0$, we find that $\Theta^{(1)}$ satisfies the following first order differential equation,

$$\frac{d\Theta^{(1)}}{dx} = \Gamma^{(2)}\Theta^{(0)} - \Theta^{(1)}TF'(U(x)) \quad , \quad (44)$$

whose solution is

$$\Theta^{(1)}(x) = \left(\Gamma^{(2)}K^{(0)}x + K^{(1)} \right) e^{-TH(x)} \quad , \quad (45)$$

where $K^{(1)}$ is an integration constant associated with the solution of the previous equation.

By collecting the slow terms of order $1/N$ in Eq. (85), one obtains the algebraic equation

$$\Pi^{(1)}(x)(e^{i\phi} - 1) = - \left[e^{i\phi}\Theta^{(1)}(0) + \Pi^{(1)}(0) \right] \frac{\mathcal{F}(U(x))}{\mathcal{F}(U(0))} \quad , \quad (46)$$

whose solution is,

$$\begin{aligned} \Pi^{(1)}(0) &= -\Theta^{(1)}(0) = -K^{(1)}e^{-TH(0)} \quad , \\ \Pi^{(1)}(x) &= -K^{(1)}e^{-TH(0)} \frac{\mathcal{F}(U(x))}{\mathcal{F}(U(0))} \quad . \end{aligned}$$

By imposing the boundary condition $\delta U^{(1)}(x=1) = \Theta^{(1)}(1) + \Pi^{(1)}(1) = 0$, it is possible to evaluate $\Gamma^{(2)}$,

$$\Theta^{(1)}(1) + \Pi^{(1)}(1) = (K^{(0)}\Gamma^{(2)} + K^{(1)})e^{-TH(1)} - K^{(1)}e^{-TH(0)} \frac{\mathcal{F}(U(1))}{\mathcal{F}(U(0))} = 0 \quad . \quad (47)$$

By again exploiting Eq. (25), we find that $\Gamma^{(2)} = 0$ and, thereby (from Eq. (45))

$$\Theta^{(1)}(x) = K^{(1)}e^{-TH(x)} \quad . \quad (48)$$

Altogether, we can conclude that the second order correction to the Floquet exponent vanishes as well, and one cannot remove the degeneracy among the eigenvectors.

5.4 Second order approximation

By assembling fast terms of order $1/N^3$ appearing in Eq. (85) and by setting $\Gamma^{(1)} = \Gamma^{(2)} = 0$, the following first order differential equation for $\Theta^{(2)}$ can be derived

$$\frac{d\Theta^{(2)}}{dx} = \Gamma^{(3)}\Theta^{(0)} + \Theta^{(2)}TF'(U(x)) \quad , \quad (49)$$

whose solution is

$$\Theta^{(2)}(x) = \left(\Gamma^{(3)}K^{(0)}x + K^{(2)} \right) e^{-TH(x)} \quad , \quad (50)$$

where $K^{(2)}$ is an integration constant associated with the solution of the previous differential equation.

Furthermore, by collecting the slow terms of order $1/N^2$, we obtain the algebraic equation,

$$\begin{aligned} \Pi^{(2)}(x)(e^{i\phi_k} - 1) &= g\alpha^2 T \Theta^{(0)}(0) \frac{e^{2i\phi_k} + 10e^{i\phi_k} + 1}{12(e^{i\phi_k} - 1)} \frac{F(U(0)) - F(U(x))}{[\mathcal{F}(U(0))]^2} \\ &- \left[e^{i\phi_k} \Theta^{(2)}(0) + \Pi^{(2)}(0) \right] \frac{\mathcal{F}(U(x))}{\mathcal{F}(U(0))} . \end{aligned}$$

By imposing that the above equation is satisfied for $x = 0$, it reduces to

$$\begin{aligned} \Pi^{(2)}(0) &= -\Theta^{(2)}(0) = -K^{(2)} e^{-TH(0)} , \\ \Pi^{(2)}(x) &= g\alpha^2 T \Theta^{(0)}(0) \frac{e^{2i\phi_k} + 10e^{i\phi_k} + 1}{12(e^{i\phi_k} - 1)^2} \frac{F(U(0)) - F(U(x))}{[\mathcal{F}(U(0))]^2} - \Theta^{(2)}(0) \frac{\mathcal{F}(U(x))}{\mathcal{F}(U(0))} . \end{aligned}$$

Finally, by imposing the boundary condition $\delta U^{(2)}(x=1) = \Theta^{(2)}(1) + \Pi^{(2)}(1) = 0$, it is possible to determine $\Gamma^{(3)}$,

$$\Gamma^{(3)} = \frac{g\alpha^2}{12} T \frac{F(U(0)) - F(U(1))}{\mathcal{F}(U(0))\mathcal{F}(U(1))} \left(\frac{6}{1 - \cos \phi_k} - 1 \right) . \quad (51)$$

Accordingly, $\Gamma^{(3)}$ is real and depends on the difference between $F(U(x=1)) \equiv F(0)$ and $F(U(x=0)) \equiv F(1)$, confirming the numerical findings in [21]. Therefore, the imaginary terms ω_i are smaller than $1/N^2$.

In the specific example of a leaky integrate-and-fire neuron the expression for $\Gamma^{(3)}$ reduces to

$$\Gamma^{(3)} = \frac{g\alpha^2}{12} T \left(2 - e^T - e^{-T} \right) \left(\frac{6}{1 - \cos \phi_k} - 1 \right) , \quad (52)$$

since, by using the equations that characterize LIF neurons, the following relation holds

$$\frac{F(U(1)) - F(U(0))}{\mathcal{F}(U(1))\mathcal{F}(U(0))} = \frac{1}{(a + \frac{g}{T})^2 e^{-T}} = \frac{1}{(\frac{1}{1-e^{-T}})^2 e^{-T}} = (e^T + e^{-T} - 2) . \quad (53)$$

All in all, Eq. (51) generalizes the expression found for the LIF model Eq. (52) [21]¹.

6 Conclusions

We have derived analytically the short-wavelength component of the Floquet spectrum of the splay solution in a finite, fully coupled, network composed of *generic* suprathreshold pulse-coupled phase-like neurons. This component is marginally stable in the thermodynamic limit and thereby requires a particular care. The analytical estimation of the long-wavelength component was previously derived in the small-coupling limit [2]. It would be nice to extend such analysis to finite coupling strength, but this is a rather problematic goal, since the eigenvalues remain

¹ In comparing with [21] one should pay attention to the different normalization used here to define μ_k in Eq. (34)

finite in the thermodynamic limit and so there is no evident smallness parameter to invoke for a safe expansion.

Our analysis has revealed that, in discontinuous velocity fields, the SW spectrum scales as $1/N^2$, and the stability is controlled by the sign of the difference between the velocity at reset and at threshold. The shape of the spectrum is otherwise *universal*, at least for a given choice of the post-synaptic potential. Our formalism could be easily implemented for any pulse shape, provided that Eq. (3) is replaced by the appropriate evolution equation. Preliminary numerical studies anyway suggest that different (e.g., purely exponential) pulses yield the same scaling behaviour, but are characterized by different Floquet spectra [27].

Moreover it is worth recalling that δ -like pulses in networks of LIF neurons give rise to a different scenario, with a finite (in)stability of the whole SW component [18]. The difference is so strong that the two scenarios cannot be reconciled even by taking the limit $\alpha \rightarrow \infty$ (zero pulsewidth) as the limits $N \rightarrow \infty$ and zero pulsewidth limit do not commute [18]. This reveals that even the *simple* construction of a general stability theory of the splay states requires some further progress.

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A Fixed-point expansion (general case)

A simple calculation shows that the splay state expression (14) can be easily obtained by first solving Eq. 4 (see also [18])

$$\tilde{P} = \frac{\alpha^2}{N} \frac{1}{(1 - e^{-\alpha T/N})} \quad , \quad \tilde{E} = \frac{T}{N} \frac{\tilde{P}}{(e^{\alpha T/N} - 1)} \quad , \quad (54)$$

where T is the period of the splay state, which must be determined self-consistently.

The $1/N$ expansion of these exact expressions leads to

$$\begin{aligned} \tilde{P} = & \frac{\alpha}{T^{(0)}} + \left[\frac{\alpha}{2} - \frac{T^{(1)}}{T^{(0)^2}} \right] \frac{\alpha}{N} + \left[\frac{\alpha^2 T^{(0)}}{12} - \frac{T^{(2)}}{T^{(0)^2}} + \frac{T^{(1)^2}}{T^{(0)^2}} \right] \frac{\alpha}{N^2} \\ & + \left[\frac{\alpha^2 T^{(1)}}{12} - \frac{T^{(3)}}{T^{(0)^2}} + 2 \frac{T^{(1)} T^{(2)}}{T^{(0)^3}} - \frac{T^{(1)^3}}{T^{(0)^4}} \right] \frac{\alpha}{N^3} + O\left(\frac{1}{N^4}\right) \quad , \end{aligned} \quad (55)$$

$$\begin{aligned} \tilde{E} = & \frac{1}{T^{(0)}} - \frac{T^{(1)}}{T^{(0)^2} N} + \left[-\frac{\alpha^2 T^{(0)}}{12} - \frac{T^{(2)}}{T^{(0)^2}} + \frac{T^{(1)^2}}{T^{(0)^3}} \right] \frac{1}{N^2} \\ & + \left[-\frac{\alpha^2}{12} T^{(1)} - \frac{T^{(3)}}{T^{(0)^2}} + 2 \frac{T^{(1)} T^{(2)}}{T^{(0)^3}} - \frac{T^{(1)^3}}{T^{(0)^4}} \right] \frac{1}{N^3} + O\left(\frac{1}{N^4}\right) \quad , \end{aligned} \quad (56)$$

$$\dot{\tilde{E}} = \frac{\alpha^2}{2N} + \frac{\alpha^2 T^{(0)}}{6N^2} + \frac{\alpha^3 T^{(1)}}{6N^3} + O\left(\frac{1}{N^4}\right) \quad , \quad (57)$$

where we have reported also the expansion of $\dot{\tilde{E}}$ that is necessary to pass from expression (13) to (17). Please notice that while the membrane potentials and the period are expanded up to $\mathcal{O}(1/N^4)$, as in (15) and (16), here we limit the expansion to $\mathcal{O}(1/N^3)$ terms, since the field variables appearing in the event-driven map are integrated over an interspike-interval (see (8)).

To proceed further, we need also to introduce the expansions of the velocity field and of its derivatives,

$$\begin{aligned} F(\tilde{u}_j) &= \overline{F}_j + \overline{F}'_j \frac{\tilde{u}_j^{(1)}}{N} + \overline{F}''_j \frac{\tilde{u}_j^{(2)}}{N^2} + \overline{F}'''_j \frac{\tilde{u}_j^{(3)}}{N^3} + \overline{F}''''_j \frac{[\tilde{u}_j^{(1)}]^2}{2N^2} + \overline{F}''''_j \frac{\tilde{u}_j^{(1)}\tilde{u}_j^{(2)}}{N^3} + \overline{F}''''_j \frac{[\tilde{u}_j^{(1)}]^3}{6N^3} + O\left(\frac{1}{N^4}\right) \\ F'(\tilde{u}_j) &= \overline{F}'_j + \overline{F}''_j \frac{\tilde{u}_j^{(1)}}{N} + \overline{F}'''_j \frac{\tilde{u}_j^{(2)}}{N^2} + \overline{F}''''_j \frac{[\tilde{u}_j^{(1)}]^2}{2N^2} + O\left(\frac{1}{N^3}\right) \\ F''(\tilde{u}_j) &= \overline{F}''_j + \overline{F}'''_j \frac{\tilde{u}_j^{(1)}}{N} + O\left(\frac{1}{N^2}\right) \end{aligned} ,$$

where the overline means that the function is computed in $\tilde{u}_j^{(0)}$, which corresponds to the infinite N limit.

By replacing the membrane potentials, the period, the self-consistent fields and the velocity field with their expansions, the event-driven map (13) can be formally rewritten for the splay state as (17) with the introduction of the following auxiliary variables

$$\mathcal{Q}^{(1)} = g + T^{(0)}\overline{F}_j \quad , \quad (58)$$

$$\mathcal{Q}^{(2)} = T^{(1)}\overline{F}_j + \left[\tilde{u}_j^{(1)} + \frac{g}{2} + \frac{\overline{F}_j}{2}T^{(0)} \right] \overline{F}'_j T^{(0)} \quad , \quad (59)$$

$$\begin{aligned} \mathcal{Q}^{(3)} &= \left[\overline{F}'_j \tilde{u}_j^{(2)} + \frac{\overline{F}''_j}{2} [\tilde{u}_j^{(1)}]^2 + g \frac{\overline{F}''_j}{2} \tilde{u}_j^{(1)} + \frac{g^2}{6} \overline{F}''_j + \left(\frac{2g}{3} \overline{F}'_j \overline{F}_j + \overline{F}_j'^2 \tilde{u}_j^{(1)} + \overline{F}_j'' \overline{F}_j \tilde{u}_j^{(1)} + \frac{g}{3} \overline{F}_j'^2 \right) \frac{T^{(0)}}{2} \right. \\ &\quad \left. + \left(\overline{F}_j'' \overline{F}_j + \overline{F}_j'^2 \right) \frac{T^{(0)^2} \overline{F}_j}{6} \right] T^{(0)} + \left[\tilde{u}_j^{(1)} + \frac{g}{2} + \overline{F}_j T^{(0)} \right] T^{(1)} \overline{F}'_j + T^{(2)} \overline{F}_j \quad , \quad (60) \end{aligned}$$

$$\begin{aligned} \mathcal{Q}^{(4)} &= \left[\overline{F}'_j \tilde{u}_j^{(3)} + \left(\tilde{u}_j^{(1)} + \frac{g}{2} \right) \overline{F}''_j \tilde{u}_j^{(2)} + \left(\frac{1}{6} [\tilde{u}_j^{(1)}]^3 + \frac{g}{4} [\tilde{u}_j^{(1)}]^2 + \frac{g^2}{6} \tilde{u}_j^{(1)} + \frac{g^3}{24} \right) \overline{F}_j'''' \right] T^{(0)} \\ &+ \left[\left(\overline{F}_j'^2 + \overline{F}_j \overline{F}_j'' \right) \frac{\tilde{u}_j^{(2)}}{2} + \left(3\overline{F}_j' \overline{F}_j'' + \overline{F}_j \overline{F}_j''' \right) \frac{[\tilde{u}_j^{(1)}]^2}{4} + g \left(2\overline{F}_j' \overline{F}_j'' + \overline{F}_j \overline{F}_j''' \right) \frac{\tilde{u}_j^{(1)}}{3} + \frac{g^2}{6} \overline{F}_j' \overline{F}_j'' + \frac{g^2}{8} \overline{F}_j \overline{F}_j''' \right] T^{(0)^2} \\ &+ \left[\left(g + 4\tilde{u}_j^{(1)} \right) \overline{F}_j'^3 + \left(8g + 16\tilde{u}_j^{(1)} \right) \overline{F}_j \overline{F}_j' \overline{F}_j'' + \left(3g + 4\tilde{u}_j^{(1)} \right) \overline{F}_j^2 \overline{F}_j''' \right] \frac{T^{(0)^3}}{24} + \left[\overline{F}_j \overline{F}_j'^3 + 4\overline{F}_j^2 \overline{F}_j' \overline{F}_j'' + \overline{F}_j^3 \overline{F}_j''' \right] \frac{T^{(0)^4}}{24} \\ &+ \left[\overline{F}_j \tilde{u}_j^{(2)} + \frac{\overline{F}_j''}{2} [\tilde{u}_j^{(1)}]^2 + g \frac{\overline{F}_j''}{2} \tilde{u}_j^{(1)} + \frac{g^2}{6} \overline{F}_j'' \right] T^{(1)} + \left[\frac{2g}{3} \overline{F}_j' \overline{F}_j + \overline{F}_j'^2 \tilde{u}_j^{(1)} + \overline{F}_j'' \overline{F}_j \tilde{u}_j^{(1)} + \frac{g}{3} \overline{F}_j'^2 \right] T^{(0)} T^{(1)} \\ &+ \frac{\overline{F}_j}{2} \left(\overline{F}_j'' \overline{F}_j + \overline{F}_j'^2 \right) T^{(0)^2} T^{(1)} + \left\{ \left(\tilde{u}_j^{(1)} + \frac{g}{2} \right) T^{(2)} + \frac{\overline{F}_j}{2} \left[T^{(1)^2} + 2T^{(0)} T^{(2)} \right] \right\} \overline{F}_j' + \overline{F}_j T^{(3)} . \end{aligned} \quad (61)$$

B Fixed-point expansion (LIF model)

In the case of the LIF neuron (see Eq. (2)), the fixed point of the event-driven map reads

$$u_{i-1} = e^{-\tau} u_i + \chi \quad , \quad (62)$$

where

$$\chi = a(1 - e^{-\tau}) + g \frac{e^{-\tau} - e^{-\alpha\tau}}{\alpha - 1} \left(E + \frac{P}{\alpha - 1} \right) - g \frac{\tau}{\alpha - 1} e^{-\alpha\tau} P. \quad (63)$$

Its solution is

$$u_j = \chi \frac{1 - e^{-NT+j\tau}}{1 - e^{-\tau}}. \quad (64)$$

By expanding Eq. (64) for $j = 0$ and for a generic j , one can derive perturbative expressions for the period T and the membrane potential, respectively. Let us start by substituting the expressions (16, 55, 56, 57) in Eqs. (63). This leads to the expansion

$$\chi = \left(a + \frac{g}{T} \right) \left[\tau - \frac{\tau^2}{2} + \frac{\tau^3}{6} - \frac{\tau^4}{24} \right] + a \frac{\tau^5}{120} + \frac{g}{T} \frac{\tau^5}{720} K(\alpha) + O(1/N^4) \quad , \quad (65)$$

where

$$K(\alpha) = \frac{360\alpha^6 - 722\alpha^5 + 363\alpha^4 + 5\alpha^2 - 12\alpha + 6}{(\alpha - 1)^2} \quad , \quad (66)$$

accounts for the dependence on the field dynamics. Now, with the help of Eqs. (16, 65) and expanding the exponential terms up to the fourth order, we obtain a closed equation for the interspike interval,

$$\begin{aligned} u_0 = 1 = & \left(a + \frac{g}{T^{(0)}} \right) \left(1 - e^{-T^{(0)}} \right) + \frac{T^{(1)}}{N} \xi(T^{(0)}) + \frac{1}{N^2} \left[T^{(2)} \xi(T^{(0)}) + T^{(1)} \mathcal{W}_1^{(2)} \right] \\ & + \frac{1}{N^3} \left[T^{(3)} \xi(T^{(0)}) + T^{(1)} \mathcal{W}_1^{(3)} + T^{(2)} \mathcal{W}_2^{(3)} \right] + \frac{1}{N^4} \left[T^{(4)} \xi(T^{(0)}) + \zeta(T^{(0)}) + T^{(1)} \mathcal{W}_1^{(4)} \right. \\ & \left. + T^{(2)} \mathcal{W}_2^{(4)} + T^{(3)} \mathcal{W}_3^{(4)} \right] \quad , \end{aligned} \quad (67)$$

where

$$\begin{aligned} \zeta(T^{(0)}) &= -(1 - e^{-T^{(0)}}) \frac{g}{120} T^{(0)^3} \left(1 - \frac{K(\alpha)}{6} \right) \quad , \\ \xi(T^{(0)}) &= \left(a + \frac{g}{T^{(0)}} \right) e^{-T^{(0)}} - \frac{g}{T^{(0)^2}} \left(1 - e^{-T^{(0)}} \right) \quad , \end{aligned}$$

while $\mathcal{W}_i^{(j)}$ identifies a term of order $1/N^j$ that is multiplied by $T^{(i)}$. Since, while proceeding from lower to higher-order terms, we find that $T^{(i)} = 0$ (for $i < 4$), it is not necessary to give the explicit expression of the $\mathcal{W}_i^{(j)}$ functions as they do not contribute at all.

One can equivalently expand u_j

$$\begin{aligned} u_j = & u_j^{(0)} + \frac{u_j^{(1)}}{N} + \frac{u_j^{(2)}}{N^2} + \frac{u_j^{(3)}}{N^3} + \frac{u_j^{(4)}}{N^4} = \left(a + \frac{g}{T^{(0)}} \right) \left(1 - e^{T^{(0)}(\frac{j}{N}-1)} \right) + \frac{T^{(1)}}{N} \varsigma(T^{(0)}) \\ & + \frac{1}{N^2} \left[T^{(2)} \varsigma(T^{(0)}) + T^{(1)} \mathcal{Z}_1^{(2)} \right] + \frac{1}{N^3} \left[T^{(3)} \varsigma(T^{(0)}) + T^{(1)} \mathcal{Z}_1^{(3)} + T^{(2)} \mathcal{Z}_2^{(3)} \right] \\ & + \frac{1}{N^4} \left[\varsigma(T^{(0)}) T^{(4)} + \sigma(T^{(0)}) + T^{(1)} \mathcal{Z}_1^{(4)} + T^{(2)} \mathcal{Z}_2^{(4)} + T^{(3)} \mathcal{Z}_3^{(4)} \right] \quad , \end{aligned} \quad (68)$$

where

$$\begin{aligned} \varsigma(T^{(0)}) &= - \left(a + \frac{g}{T^{(0)}} \right) e^{T^{(0)}(\frac{j}{N}-1)} \left(\frac{j}{N} - 1 \right) - \frac{g}{T^{(0)^2}} \left[1 - e^{T^{(0)}(\frac{j}{N}-1)} \right] \quad , \\ \sigma(T^{(0)}) &= \frac{g}{120} (1 - e^{T^{(0)}(\frac{j}{N}-1)}) T^{(0)^3} \left(\frac{K}{6} - 1 \right) \quad , \end{aligned}$$

while we do not provide explicit expressions for $\mathcal{Z}_i^{(j)}$ as they turn out to be irrelevant.

Now we are in the position to analyse the different orders.

B.1 Zeroth Order

By assembling the terms of order 1 in Eq. (67), we obtain

$$\left(a + \frac{g}{T^{(0)}}\right) (1 - e^{-T^{(0)}}) = 1. \quad (69)$$

This is an implicit definition of the asymptotic interspike time $T^{(0)}$

$$T^{(0)} = \ln \left(\frac{aT^{(0)} + g}{T^{(0)}(a-1) + g} \right). \quad (70)$$

Analogously, we can find an explicit equation for the membrane potential by assembling the terms of order 1 in Eq. (68)

$$u_j^{(0)} = \left(a + \frac{g}{T^{(0)}}\right) \left[1 - e^{T^{(0)}\left(\frac{j}{N}-1\right)}\right] \quad . \quad (71)$$

In the thermodynamic limit the solution for $u_j^{(0)}$ becomes

$$U^{(0)}(x) = \left(a + \frac{g}{T^{(0)}}\right) \left[1 - e^{T^{(0)}(x-1)}\right] \quad , \quad (72)$$

which coincides with Eq. (21) with $F = a - U^{(0)}$.

B.2 From first to third order

By separately assembling the terms of order $1/N^i$ (for $i = 1, 2, 3$) in Eq. (67), we obtain

$$T^{(i)}\xi(T^{(0)}) = 0 \quad , \quad (73)$$

which implies that $T^{(i)} = 0$ since $\xi \neq 0$. Moreover, by assembling the terms of order $1/N^i$ in Eq. (68), we obtain

$$u_j^{(1)} = \varsigma(T^{(0)})T^{(i)} \quad , \quad (74)$$

which thereby implies that first, second and third order corrections vanish also for the membrane potential.

B.3 Fourth Order

The order which reveals a different scenario is the fourth one. By assembling the terms of order $1/N^4$ in Eq. (67) we obtain

$$T^{(4)} = -\frac{\zeta(T^{(0)})}{\xi(T^{(0)})} \quad , \quad (75)$$

whose explicit expression is reported in Eq. (5). By analogously assembling the terms of order $1/N^4$ in Eq. (68), we obtain

$$u_j^{(4)} = \varsigma(T^{(0)})T^{(4)} + \sigma(T^{(0)}) \quad , \quad (76)$$

which becomes, in the thermodynamic limit,

$$\begin{aligned} U^{(4)}(x) = & -\left(a + \frac{g}{T^{(0)}}\right) T^{(4)} e^{T^{(0)}(x-1)}(x-1) - g \frac{T^{(4)}}{T^{(0)2}} \left[1 - e^{T^{(0)}(x-1)}\right] \\ & + \frac{g}{120} (1 - e^{T^{(0)}(x-1)}) T^{(0)3} \left(\frac{K(\alpha)}{6} - 1\right) \quad . \end{aligned}$$

C Expansion in tangent space around the fixed point

C.1 Introduction

The first equations of the tangent map can be determined by differentiating Eq. (4) and thereby expanding in powers of τ (this is equivalent to expanding in powers of $1/N$, as the dependence of τ on N would only generate higher order terms),

$$\delta P_{n+1} = \left(1 - \alpha\tau + \frac{\alpha^2}{2}\tau^2 - \frac{\alpha^3}{6}\tau^3 + \frac{\alpha^4}{24}\tau^4\right) \delta P_n + \tilde{P} \left(-\alpha + \alpha^2\tau - \frac{\alpha^3}{2}\tau^2 + \frac{\alpha^4}{6}\tau^3\right) \delta\tau_n, \quad (77)$$

$$\begin{aligned} \delta E_{n+1} = & \left(1 - \alpha\tau + \frac{\alpha^2}{2}\tau^2 - \frac{\alpha^3}{6}\tau^3 + \frac{\alpha^4}{24}\tau^4\right) \delta E_n + \left(\tau - \alpha\tau^2 + \frac{\alpha^2}{2}\tau^3 - \frac{\alpha^3}{6}\tau^4\right) \delta P_n \\ & + \left[-\alpha\tilde{E} \left(1 - \alpha\tau + \frac{\alpha^2}{2}\tau^2 - \frac{\alpha^3}{6}\tau^3 + \frac{\alpha^4}{24}\tau^4\right) + \tilde{P} \left(1 - 2\alpha\tau + \frac{3}{2}\alpha^2\tau^2 - \frac{2}{3}\alpha^3\tau^3 + \frac{5}{24}\alpha^4\tau^4\right)\right] \delta\tau_n, \end{aligned} \quad (78)$$

where the dependence of τ on n has been dropped, since we are considering a linearization around the splay state.

By further differentiating Eq. (13) around the fixed point solution, one obtains

$$\begin{aligned} \delta u_{n+1,i-1} = & \delta u_{n,i} + (\bar{F}'_i \delta u_{n,i} + g \delta E_n) \tau + [\bar{F}''_i \bar{\mathcal{F}}_i \delta u_{n,i} + \bar{F}'_i (\bar{F}'_i \delta u_{n,i} + g \delta E_n) + g \delta \dot{E}_n] \frac{\tau^2}{2} \\ & + \left\{ 2g\bar{F}''_i \bar{\mathcal{F}}_i \delta E_n + [\bar{F}'''_i \bar{\mathcal{F}}_i^2 + \bar{F}''_i (4\bar{F}'_i \bar{\mathcal{F}}_i + g\dot{E})] \delta u_{n,i} + \bar{F}'_i (\bar{F}'_i \delta u_{n,i} + g \delta E_n) + g\bar{F}'_i \delta \dot{E}_n \right. \\ & \left. - g\alpha (\delta P_n + \delta \dot{E}_n) \right\} \frac{\tau^3}{6} + \left\{ \bar{\mathcal{F}}_i + [\bar{F}'_i \bar{\mathcal{F}}_i + g\dot{E}] \tau + [\bar{F}''_i \bar{\mathcal{F}}_i^2 + (\bar{F}'_i)^2 \bar{\mathcal{F}}_i + g\bar{F}'_i \dot{E} - g\alpha (\dot{E} + \tilde{P})] \frac{\tau^2}{2} \right. \\ & \left. + [\bar{F}'_i)^3 \bar{\mathcal{F}}_i + 4\bar{F}'_i \bar{F}''_i \bar{\mathcal{F}}_i^2 + \bar{F}''_i \bar{\mathcal{F}}_i^3 + g\alpha(2\alpha - \bar{F}'_i) \tilde{P}] \frac{\tau^3}{6} \right\} \delta\tau_n \end{aligned} \quad (79)$$

Finally, $\delta\tau_n$ can be determined by differentiating Eq. (13) for $i = 1$

$$\begin{aligned} \delta\tau_n = & -\frac{\delta E_n}{\bar{\mathcal{F}}_1} \left\{ g\tau - g \left[\frac{1}{2}(\bar{F}'_1 + \alpha) + \frac{1}{\bar{\mathcal{F}}_1} g\dot{E} \right] \tau^2 + \frac{g}{\bar{\mathcal{F}}_1} \left[g\alpha \left(\dot{E} + \frac{\tilde{P}}{2} \right) + \frac{\bar{\mathcal{F}}_1}{6} \left(\alpha^2 - 5\bar{F}'_1{}^2 - \bar{F}'_1 \bar{\mathcal{F}}_1 + 2\alpha\bar{F}'_1 \right) \right. \right. \\ & \left. \left. + \frac{1}{\bar{\mathcal{F}}_1} (\bar{F}'_1 \bar{\mathcal{F}}_1 + g\dot{E})^2 \right] \tau^3 \right\} - \frac{\delta P_n}{\bar{\mathcal{F}}_1} \left\{ \frac{g}{2}\tau^2 - \frac{g}{2}\tau^3 \left[\frac{2}{3}(\alpha + \bar{F}'_1) + \frac{1}{\bar{\mathcal{F}}_1} g\dot{E} \right] \right\} - \frac{\delta u_{n,1}}{\bar{\mathcal{F}}_1} \left\{ 1 - \frac{g}{\bar{\mathcal{F}}_1} \tau \dot{E} \right. \\ & \left. + \tau^2 \left[\frac{g\alpha}{2\bar{\mathcal{F}}_1} (\dot{E} + \tilde{P}) + \frac{1}{\bar{\mathcal{F}}_1^2} (\bar{F}'_1 \bar{\mathcal{F}}_1 + g\dot{E})^2 - \bar{F}'_1{}^2 - \frac{3}{2} \frac{g}{\bar{\mathcal{F}}_1} \bar{F}'_1 \dot{E} \right] - \tau^3 \left[\frac{1}{3\bar{\mathcal{F}}_1} g\alpha (\alpha + \bar{F}'_1) \tilde{P} \right. \right. \\ & \left. \left. + \frac{1}{\bar{\mathcal{F}}_1^2} g\alpha \dot{E} \left(\frac{1}{2} \bar{F}'_1 \bar{\mathcal{F}}_1 + g\tilde{P} \right) + \frac{1}{\bar{\mathcal{F}}_1^3} g^2 \dot{E}^3 + \frac{1}{\bar{\mathcal{F}}_1^2} g^2 \dot{E}^2 (\alpha + \bar{F}'_1) - \frac{2}{3} g\bar{F}'_1 \dot{E} \right] \right\}. \end{aligned} \quad (80)$$

In order to find the Floquet eigenvalue μ_k , one should substitute the Ansätze (33) into Eqs. (77,78). This allows to find explicit expressions for δP and δE as a function of μ_k , τ and $\delta\tau$, namely

$$\delta P = -\frac{\alpha^2}{\mu_k - 1} \left[1 - \frac{\alpha\tau}{2} \frac{\mu_k + 1}{\mu_k - 1} + \alpha^2 \tau^2 M_k - \frac{\alpha^3 \tau^3}{2} \frac{\mu_k (\mu_k + 1)}{(\mu_k - 1)^3} \right] \frac{\delta\tau}{T}, \quad (81)$$

$$\delta E = -\frac{\alpha}{\mu_k - 1} \left[\frac{\alpha\tau}{2} \frac{(\mu_k + 1)}{\mu_k - 1} - 2\alpha^2 \tau^2 M_k - \frac{3\alpha^3 \tau^3}{2} \frac{\mu_k (\mu_k + 1)}{(\mu_k - 1)^3} \right] \frac{\delta\tau}{T}, \quad (82)$$

where we have introduced the shorthand notation

$$M_k = \frac{\mu_k^2 + 10\mu_k + 1}{12(\mu_k - 1)^2} . \quad (83)$$

By substituting δP and δE , as given by (81) and (82), into Eq. (80), we can express $\delta\tau$ directly in terms of δu_1

$$\delta\tau = - \left\{ \bar{\mathcal{F}}_1 + \frac{g\alpha^2}{T} M_k \tau^2 - \frac{g\alpha^2}{T} \left[\frac{\bar{\mathcal{F}}_1'}{12} \frac{(\mu_k + 5)}{(\mu_k - 1)^2} + \alpha \frac{(\mu_k + 1)}{(\mu_k - 1)^3} \right] \mu_k \tau^3 \right\} \frac{\delta u_1}{\bar{\mathcal{F}}_1^2}, \quad (84)$$

where we exploited the equality $\bar{\mathcal{F}}_i \equiv \bar{F}_i + \frac{g}{T}$ which follows from the fact that in the thermodynamic limit $\bar{E} = \frac{1}{T}$.

By inserting the expressions in Eqs. (81, 82, 84) into Eq. (79), we find a single equation for the eigenvalues and eigenvectors,

$$\begin{aligned} \mu_k \delta u_{i-1} = & \left\{ 1 + \bar{F}_i' \tau + \left[\bar{F}_i'' \bar{\mathcal{F}}_i + \bar{F}_i'^2 \right] \frac{\tau^2}{2} + \left[\bar{F}_i''' \bar{\mathcal{F}}_i^2 + 4\bar{F}_i' \bar{F}_i'' \bar{\mathcal{F}}_i + \bar{F}_i'^3 \right] \frac{\tau^3}{6} \right\} \delta u_i \\ & - \left\{ \bar{\mathcal{F}}_i + \bar{F}_i' \bar{\mathcal{F}}_i \tau + \left[\frac{\bar{F}_i''}{2} \bar{\mathcal{F}}_i^2 + \frac{\bar{F}_i'^2}{2} \bar{\mathcal{F}}_i + \frac{g\alpha^2}{T} M_k \left(\frac{\bar{\mathcal{F}}_i}{\bar{\mathcal{F}}_1} - 1 \right) \right] \tau^2 + \left[\frac{\bar{F}_i'^3}{6} \bar{\mathcal{F}}_i + \frac{2}{3} \bar{F}_i' \bar{F}_i'' \bar{\mathcal{F}}_i^2 + \frac{\bar{F}_i'''}{6} \bar{\mathcal{F}}_i^3 \right. \right. \\ & \left. \left. + \frac{g\alpha^2}{T} \frac{5\mu_k + 1}{12(\mu_k - 1)^2} \left(\frac{\bar{\mathcal{F}}_i}{\bar{\mathcal{F}}_1} \bar{F}_1' - \bar{F}_i' \right) + \frac{g\alpha^3}{T} \frac{\mu_k(\mu_k + 1)}{(\mu_k - 1)^3} \left(1 - \frac{\bar{\mathcal{F}}_i}{\bar{\mathcal{F}}_1} \right) + \frac{g\alpha^2}{T} \frac{\bar{\mathcal{F}}_i}{\bar{\mathcal{F}}_1} (\bar{F}_i' - \bar{F}_1') M_k \right] \tau^3 \right\} \frac{\delta u_1}{\bar{\mathcal{F}}_1}. \end{aligned}$$

By now substituting the μ_k expansion (34) and retaining the leading terms, we obtain Eq. (35).

C.2 $N \rightarrow \infty$ limit

Once the continuous variables (38) have been introduced, it is necessary to estimate $U(1/N)$ and $\delta U(1/N)$, by expanding such variables around zero. By inserting the resulting expansion for $U(1/N)$ into the expressions for \bar{F}_1 and $\bar{\mathcal{F}}_1$, we obtain, respectively

$$\begin{aligned} F\left(\frac{1}{N}\right) = & F(U(0)) + \frac{F'(U(0))}{N} \frac{dU}{dx} \Big|_0 + \frac{F'(U(0))}{2N^2} \frac{d^2U}{dx^2} \Big|_0 + \frac{F''(U(0))}{2} \left(\frac{1}{N} \frac{dU}{dx} \Big|_0 \right)^2 \\ & + \frac{F''(U(0))}{2N^3} \frac{dU}{dx} \Big|_0 \frac{d^2U}{dx^2} \Big|_0 + \frac{F'''(U(0))}{6} \left(\frac{1}{N} \frac{dU}{dx} \Big|_0 \right)^3 + \frac{F'(U(0))}{6N^3} \frac{d^3U}{dx^3} \Big|_0 + O\left(\frac{1}{N^4}\right), \end{aligned}$$

$$\begin{aligned} \frac{1}{\mathcal{F}(U(\frac{1}{N}))} \equiv & \frac{1}{F(U(\frac{1}{N})) + \frac{g}{T}} = \frac{1}{\mathcal{F}(U(0))} + \frac{1}{N} \frac{TF'(U(0))}{\mathcal{F}(U(0))} + \frac{1}{2N^2} \frac{[TF'(U(0))]^2}{\mathcal{F}(U(0))} - \frac{1}{2N^2} F''(U(0)) T^2 \\ & + \frac{T^3}{6N^3} \left[F'''(U(0)) \mathcal{F}(U(0)) - 2F'(U(0)) F''(U(0)) - \frac{g}{T^2} \left(\frac{F'(U(0))}{\mathcal{F}(U(0))} \right)^2 - 5 \frac{[F'(U(0))]^3}{\mathcal{F}(U(0))} + 6 \frac{F'(U(0))}{\mathcal{F}(U(0))} \right] + O\left(\frac{1}{N^4}\right). \end{aligned}$$

An analogous procedure for $\delta U(1/N)$ leads to

$$\begin{aligned} \delta U(1/N) = & e^{i\phi_k} \left[\Theta^{(0)}\left(\frac{1}{N}\right) + \frac{\Theta^{(1)}(\frac{1}{N})}{N} + \frac{\Theta^{(2)}(\frac{1}{N})}{N^2} \right] + \Pi^{(0)}\left(\frac{1}{N}\right) + \frac{\Pi^{(1)}(\frac{1}{N})}{N} + \frac{\Pi^{(2)}(\frac{1}{N})}{N^2} + O\left(\frac{1}{N^3}\right) \\ \equiv & C^{(0)} + \frac{C^{(1)}}{N} + \frac{C^{(2)}}{N^2} + O\left(\frac{1}{N^3}\right), \end{aligned}$$

where $C^{(0)}$, $C^{(1)}$, $C^{(2)}$ are defined according to the following equations

$$\begin{aligned} C^{(0)} &= e^{i\phi_k} \Theta^{(0)}(0) + \Pi^{(0)}(0) \quad , \\ C^{(1)} &= e^{i\phi_k} \left[\frac{d\Theta^{(0)}}{dx} \Big|_0 + \Theta^{(1)}(0) \right] + \frac{d\Pi^{(0)}}{dx} \Big|_0 + \Pi^{(1)}(0) \quad , \\ C^{(2)} &= e^{i\phi_k} \left[\frac{1}{2} \frac{d^2\Theta^{(0)}}{dx^2} \Big|_0 + \frac{d\Theta^{(1)}}{dx} \Big|_0 + \Theta^{(2)}(0) \right] + \frac{1}{2} \frac{d^2\Pi^{(0)}}{dx^2} \Big|_0 + \frac{d\Pi^{(1)}}{dx} \Big|_0 + \Pi^{(2)}(0) . \end{aligned}$$

We now expand $\delta U(1/N)$ up to the order $O\left(\frac{1}{N^3}\right)$, thus neglecting higher orders, because they contribute to the definition of Π variable and we need terms at least of order $O\left(\frac{1}{N^3}\right)$ (one order lower than needed to define Θ). By inserting the Ansatz (36) and the previous expansions in Eq. (35), we finally obtain a closed equation for the eigenvalues and eigenvectors,

$$\begin{aligned} e^{i\phi_k} & \left\{ \Pi^{(0)} + \left[\Pi^{(1)} - \Pi^{(0)'} + \Pi^{(0)} \Gamma^{(1)} \right] \frac{1}{N} + \left[\frac{\Pi^{(0)''}}{2} - \Pi^{(1)'} + \Pi^{(2)} - \Pi^{(0)'} \Gamma^{(1)} + \Pi^{(1)} \Gamma^{(1)} \right. \right. \\ & + \Pi^{(0)} \Gamma^{(2)} \left. \right] \frac{1}{N^2} + \left[\frac{\Pi^{(1)''}}{2} - \frac{\Pi^{(0)'''}{6}}{6} - \Pi^{(2)'} + \Pi^{(3)} + \frac{\Pi^{(0)''} \Gamma^{(1)}}{2} - \Pi^{(1)'} \Gamma^{(1)} + \Pi^{(2)} \Gamma^{(1)} \right. \\ & - \Pi^{(0)'} \Gamma^{(2)} + \Pi^{(1)} \Gamma^{(2)} + \Pi^{(0)} \Gamma^{(3)} \left. \right] \frac{1}{N^3} \left. \right\} - \Pi^{(0)} - \left[\Pi^{(1)} + \Pi^{(0)} A^{(1)} \right] \frac{1}{N} - \left[\Pi^{(2)} + \Pi^{(0)} A^{(2)} \right. \\ & + \Pi^{(1)} A^{(1)} \left. \right] \frac{1}{N^2} - \left[\Pi^{(3)} + \Pi^{(0)} A^{(3)} + \Pi^{(1)} A^{(2)} + \Pi^{(2)} A^{(1)} \right] \frac{1}{N^3} + C^{(0)} B^{(0)} + \left[C^{(0)} B^{(1)} \right. \\ & + C^{(1)} B^{(0)} \left. \right] \frac{1}{N} + \left[C^{(0)} B^{(2)} + C^{(1)} B^{(1)} + C^{(2)} B^{(0)} \right] \frac{1}{N^2} + \frac{\mathcal{B}}{N^3} = e^{i\phi_k} \left\{ \left[\Theta^{(0)'} - \Theta^{(0)} \Gamma^{(1)} \right. \right. \\ & + \Theta^{(0)} A^{(1)} \left. \right] \frac{1}{N} + \left[\left(\Theta^{(0)} A^{(2)} - \frac{\Theta^{(0)''}}{2} \right) + \Theta^{(1)'} + \Theta^{(0)'} \Gamma^{(1)} - \Theta^{(1)} \Gamma^{(1)} - \Theta^{(0)} \Gamma^{(2)} \right. \\ & + \Theta^{(1)} A^{(1)} \left. \right] \frac{1}{N^2} + \left[\left(\frac{-\Theta^{(1)''}}{2} + \Theta^{(1)} A^{(2)} \right) + \left(\frac{\Theta^{(0)'''}{6}}{6} + \Theta^{(0)} A^{(3)} \right) + \Theta^{(2)'} - \frac{\Theta^{(0)''} \Gamma^{(1)}}{2} \right. \\ & + \Theta^{(1)'} \Gamma^{(1)} - \Theta^{(2)} \Gamma^{(1)} + \Theta^{(0)'} \Gamma^{(2)} - \Theta^{(1)} \Gamma^{(2)} - \Theta^{(0)} \Gamma^{(3)} + \Theta^{(2)} A^{(1)} \left. \right] \frac{1}{N^3} \left. \right\} , \end{aligned} \quad (85)$$

where we have introduced the shorthand notation \mathcal{B} in order to characterize a term of order $O\left(\frac{1}{N^3}\right)$, whose explicit expression is not necessary, since it turns out to contribute to the definition of the Π variable, and it is therefore one order beyond what we need. Moreover, notice that the terms appearing within round brackets in the rhs of the above equation can be shown to be zero, due to exact algebraic cancellations that emerge from the solution of the equation order by order. Finally,

$$\begin{aligned} A^{(1)}(U(x)) &= TF'(U(x)), & A^{(2)}(U(x)) &= \frac{T^2}{2} \left\{ F''(U(x)) \mathcal{F}(U(x)) + [F'(U(x))]^2 \right\} \quad , \\ A^{(3)}(U(x)) &= \frac{T^3}{6} \left\{ F'''(U(x)) [\mathcal{F}(U(x))]^2 + 4F'(U(x)) F''(U(x)) \mathcal{F}(U(x)) + [F'(U(x))]^3 \right\} \quad , \\ B^{(0)}(U(x)) &= \frac{\mathcal{F}(U(x))}{\mathcal{F}(U(0))}, & B^{(1)}(U(x)) &= \left[TF'(U(x)) + TF'(U(0)) \right] \frac{\mathcal{F}(U(x))}{\mathcal{F}(U(0))} \quad , \end{aligned}$$

$$\begin{aligned} B^{(2)}(U(x)) &= T^2 \left\{ -\frac{g}{T} \alpha^2 \frac{e^{2i\phi_k} + 10e^{i\phi_k} + 1}{12(e^{i\phi_k} - 1)^2} \frac{F(U(0)) - F(U(x))}{[\mathcal{F}(U(0))]^2} + \frac{F''(U(x))}{2} \frac{[\mathcal{F}(U(x))]^2}{\mathcal{F}(U(0))} + \right. \\ & \left. \frac{[F'(U(x))]^2}{2} \frac{\mathcal{F}(U(x))}{\mathcal{F}(U(0))} \right\} + \left\{ TF'(U(x)) TF'(U(0)) + \frac{[TF'(U(0))]^2}{2} \right\} \frac{\mathcal{F}(U(x))}{\mathcal{F}(U(0))} - \frac{T^2}{2} F''(U(0)) \mathcal{F}(U(x)) \quad . \end{aligned}$$

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